J. Fluid Mech. (1997), *vol.* 336, *pp.* 123–150

The role of angular momentum in the magnetic damping of turbulence

By P. A. DAVIDSON
Department of Engineering, University of Cambridge, Trumpington Street, Cambridge, CB2 1PZ, UK

(Received 21 February 1996 and in revised form 17 October 1996)

Landau & Lifshitz showed that Kolmogorov's $E \sim t^{-10/7}$ law for the decay of isotropic turbulence rests on just two physical ideas: (*a*) the conservation of angular momentum, as expressed by Loitsyansky's integral; and (*b*) the removal of energy from the large scales via the energy cascade. Both Kolmogorov's original analysis and Landau & Lifshitz's reinterpretation in terms of angular momentum are now known to be flawed. The existence of long-range velocity correlations means that Loitsyansky's integral is not an exact representation of angular momentum, nor is it strictly conserved. However, in practice the long-range velocity correlations are weak and Loitsyansky's integral is almost constant, so that the Kolmogorov/Landau model provides a surprisingly simple and robust description of the decay. In this paper we redevelop these ideas in the context of MHD turbulence. We take advantage of the fact that the angular momentum of a fluid moving in a uniform magnetic field has particularly simple properties. Specifically, the component parallel to the magnetic field is conserved while the normal components decay exponentially on a time scale of $\tau = \rho/\sigma \boldsymbol{B}^2$ We show that the counterpart of Loitsyansky's integral for MHD turbulence is $\int \boldsymbol{x}_\perp^2 Q_\perp \,\mathrm{d}\boldsymbol{x}$, where Q_{ij} is the velocity correlation. When the long-range correlations are weak this integral is conserved. This provides an estimate of the rate of decay of energy. At low values of magnetic field we recover Kolmogorov's law. At high values we find $E \sim t^{-1/2}$, which is a result derived earlier by Moffatt. We also show that $\int \boldsymbol{x}_\perp^2 Q_\parallel \,\mathrm{d}\boldsymbol{x}$ decays exponentially on a time scale of τ. We interpret these results in terms of the behaviour of isolated vortices orientated normal and parallel to the magnetic field.

1. Introduction

In a previous paper (Davidson 1995) the author noted that the angular momentum of a conducting fluid moving in a uniform magnetic field behaves in a particularly simple way. Specifically, when the fluid is contained in a sphere, and it is permissible to ignore shear stresses on the boundary, then the components of global angular momentum parallel and perpendicular to the magnetic field, $\boldsymbol{B}$, behave as

$$\boldsymbol{H}_\parallel(t) = \text{const.}, \tag{1.1}$$

$$\boldsymbol{H}_\perp(t) = \boldsymbol{H}_\perp(0)\exp[-t/4\tau], \tag{1.2}$$

where

$$\tau^{-1} = \sigma B^2/\rho. \tag{1.3}$$

Here σ is the electrical conductivity of the fluid, ρ is its density, and $\boldsymbol{H}$ is defined as

$$\boldsymbol{H}(t) = \int \boldsymbol{x} \times \boldsymbol{u}\,\mathrm{d}V.$$

The derivation of (1.1) and (1.2) relies on the assumption that the magnetic Reynolds number, $Re_m = \mu\sigma ul$, is small; a condition almost invariably satisfied both in the laboratory and in industrial applications.

Now a knowledge of the behaviour of $\boldsymbol{H}(t)$ can be useful in the context of decaying turbulent flows. For example, one of the earliest predictions of the decay of conventional isotropic turbulence, originally due to Kolmogorov but later reinterpreted by Landau & Lifshitz, rests on just two physical ideas: (*a*) the conservation of angular momentum, as expressed by Loitsyansky's integral; and (*b*) the removal of energy from the large scales via the energy cascade (see, for example, Landau & Lifshitz 1959 and Lesieur 1987). Now Kolmogorov's original analysis, and Landau & Lifshitz reinterpretation in terms of angular momentum, are both known to be flawed. (Loitsyansky's integral is not an exact representation of the angular momentum, nor is it strictly conserved during the decay.) Nevertheless, the Kolmogorov/Landau model still provides a surprisingly robust picture of the decay of isotropic turbulence (see §6). Indeed subsequent research, based largely on intricate, phenomenological closure models, has produced only a modest improvement in the prediction of the decay of turbulent kinetic energy, E. (Recent closure models predict $E \sim t^{-1.38}$ while Kolmogorov's law gives $E \sim t^{-1.43}$ (Lesieur 1987). Experiments indicate $E \sim t^{-1.25}$–$t^{-1.36}$).

MHD turbulence, of course, behaves quite differently from conventional turbulence. The Lorentz forces induced by a uniform magnetic field not only accelerate the decay of kinetic energy, via Joule dissipation, but also create an anisotropic eddy structure, elongating the vortices in the direction of the magnetic field (see, for example, Alemany *et al.* 1979). It is intriguing to ask whether Kolmogorov's and Landau's original analysis can be adapted, with the aid of (1.1) and (1.2), to shed light on the nature of decaying MHD turbulence. This question lies at the heart of this paper.

We start, however, by taking a step back. As a prelude to our analysis of decaying turbulent flow, we examine the influence of a uniform magnetic field on single isolated vortices, aligned parallel and perpendicular to the magnetic field. These simple model flows provide the essential physical insight which allows us to interpret our results for turbulent flow. In particular, we shall see that (1.1) and (1.2) impose strong constraints on the manner in which isolated vortices may evolve. Vortices orientated normal to $\boldsymbol{B}$ lose their angular momentum by developing thin interwoven layers of oppositely signed vorticity, while those aligned with the magnetic field preserve their angular momentum, despite Joule dissipation, by diffusing along the magnetic field lines. Their energy then decays as $(t/\tau)^{-1/2}$. We shall see that both of these features manifest themselves in more complex turbulent flows.

The structure of the paper is as follows. We start, in §2, by establishing the governing equations and listing a number of simplifying assumptions (high Reynolds number, low magnetic Reynolds number). Note that we place no restriction on the size of the interaction parameter. Subsequently, in §3, we review the results of Moffatt (1967), Sommeria & Moreau (1982), and Davidson (1995). These papers establish the behaviour of vorticity and angular momentum in a uniform magnetic field and provide the starting point for the analysis of §§4–6. Next, we examine the behaviour of simple isolated vortices aligned parallel and perpendicular to the magnetic field. These simple flows provide an insight into the altogether more complex problem of MHD turbulence. Finally, the difficult problem of decaying turbulence is discussed in §§6–9. Here we adapt Kolmogorov's original analysis, as reinterpreted by Landau & Lifshitz (1959), to MHD turbulence. This allows us to estimate the rate of decay of the turbulence energy, and the rate of generation of anisotropy as measured by the moments of the velocity correlations normal and parallel to the magnetic field. Note that our analysis is valid for arbitrary values of $\boldsymbol{B}$ and so covers the nonlinear case of weak or moderate interaction parameters. We start, then, by stating the governing equations of motion.

2. Governing equations

We are concerned with electrically conducting incompressible fluids: in effect, liquid metals. Suppose that our liquid metal occupies a domain V which is infinite in extent or else bounded by an electrically insulating surface S. We shall use both Cartesian (x, y, z) and cylindrical-polar (r, θ, z) coordinates. In either case we assume that there is a uniform magnetic field, $\boldsymbol{B}$, imposed in the z-direction. Let the velocity field be $\boldsymbol{u}$, the current density be $\boldsymbol{J}$, and a characteristic length scale be l. (In a turbulent flow, for example, l might be the integral scale of the turbulence.)

It is well known that a static magnetic field tends to damp out movement in a conducting fluid on a time scale of $\tau = \rho/\sigma B^2$ (see, for example, Shercliff 1965). In effect, this follows directly from Ohm's law. If we ignore the contribution to $\boldsymbol{J}$ from any induced electrostatic field then $\boldsymbol{J} \sim \sigma \boldsymbol{u} \times \boldsymbol{B}$, so that the Lorentz force per unit mass is $\boldsymbol{F} \sim -\boldsymbol{u}/\tau$. Pressure forces apart, individual fluid particles then decelerate on a time scale of τ. The ratio of τ to the characteristic advection time, l/u, gives the interaction parameter

$$N = \sigma B^2 l/\rho u. \tag{2.1}$$

Typically, N is indicative of the relative sizes of the magnetic and inertial forces. When N is large, the nonlinear advection term in the equation of motion may be neglected. This case has been studied by Moffatt (1967). Here we make no particular assumption aboiut the size of N, although we have in mind the case where N is moderate or large so that the magnetic field is important. We shall, however, place restrictions on the size of the Reynolds number and magnetic Reynolds number. We take the former to be large, so that large-scale motions (but not small-scale turbulence) may be treated as inviscid, while the latter is assumed to be small, so that perturbations to the applied magnetic field may be neglected. The second assumption allows us to write Ohm's law in the form

$$\boldsymbol{J} = \sigma(-\nabla \Phi + \boldsymbol{u} \times \boldsymbol{B}), \tag{2.2}$$

where Φ is the induced electrostatic potential and $\boldsymbol{B}$ is the uniform imposed magnetic field. If $\boldsymbol{u}$ is known then Φ and $\boldsymbol{J}$ are determined by the divergence and curl of (2.2) respectively:

$$\nabla^2 \Phi = \boldsymbol{B} \cdot \boldsymbol{\omega}, \tag{2.3}$$

$$\nabla \times \boldsymbol{J} = \sigma \boldsymbol{B} \cdot \nabla \boldsymbol{u}, \tag{2.4}$$

where $\boldsymbol{\omega}$ is the vorticity field. Note that $\boldsymbol{J}$, and hence the Lorentz force, is linear in $\boldsymbol{u}$ and disappears only when the motion is uniform along the field lines. The Lorentz force is

$$\boldsymbol{F} = \frac{\boldsymbol{J} \times \boldsymbol{B}}{\rho} = -\frac{\boldsymbol{u}_\perp}{\tau} + \frac{\sigma(\boldsymbol{B} \times \nabla \Phi)}{\rho} \tag{2.5}$$

which yields

$$\nabla^2(\nabla \times \boldsymbol{F}) = -\frac{1}{\tau}\frac{\partial^2 \boldsymbol{\omega}}{\partial z^2}. \tag{2.6}$$

The curl of the Lorentz force can therefore be written as

$$\nabla \times \boldsymbol{F} = -\frac{1}{\tau}\nabla^{-2}[\partial^2\boldsymbol{\omega}/\partial z^2] = -\frac{1}{\tau}\frac{\partial^2 \boldsymbol{a}}{\partial z^2} \tag{2.7}$$

where $\boldsymbol{a}$ is the solenoidal vector potential for $\boldsymbol{u}$ and the operator ∇^{-2} is defined via the Biot–Savart law. (For finite domains care must be taken in the definition of ∇^{-2} and $\boldsymbol{a}$. Here we follow the convention of Batchelor (1967, p. 86).) In a similar fashion (see Sommeria & Moreau 1982), the Lorentz force may be written as

$$\boldsymbol{F} = -\frac{1}{\tau}\nabla^{-2}[\partial^2\boldsymbol{u}/\partial z^2] + \nabla\phi, \tag{2.8}$$

where the gradient term merely augments the fluid pressure.

Since Re has been assumed to be large, we shall neglect viscosity during our treatment of isolated vortices, although it must, of course, be reinstated in our discussion of turbulence. The governing equation for vorticity is therefore

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} - \frac{1}{\tau}\nabla^{-2}[\partial^2\boldsymbol{\omega}/\partial z^2] \tag{2.9}$$

while that for linear momentum is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla(p^*/\rho) - \frac{1}{\tau}\nabla^{-2}[\partial^2\boldsymbol{u}/\partial z^2], \tag{2.10}$$

where p^* is the augmented pressure. Finally, we can construct a mechanical energy equation from the inviscid equation of motion. Noting that

$$\boldsymbol{F}\cdot\boldsymbol{u} = -\boldsymbol{J}^2/\rho\sigma - \nabla\cdot[\Phi\boldsymbol{J}/\rho],$$

and taking the dot product of $\boldsymbol{u}$ with (2.10), this takes the form

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{\rho\sigma}\int \boldsymbol{J}^2\,\mathrm{d}V = -D, \tag{2.11}$$

where E is the global kinetic per unit mass. Clearly, kinetic energy is continually dissipated as long as $\boldsymbol{J}$ is non-zero. Thus the phrase 'magnetic damping' is frequently employed to describe such flows. Note that D, the dissipation integral, is zero only if $\boldsymbol{u}$ is independent of z. This follows from (2.4).

3. Previous studies of magnetic damping

There is a wealth of literature on the magnetic damping of vortical flows, particularly in the context of MHD turbulence. We shall postpone our discussion of the experimental and numerical studies of turbulence until §6. Here we restrict ourselves to three theoretical papers which are central to the subsequent discussion. The first is that of Moffatt (1967).

Moffatt examined the decay of turbulence in a strong magnetic field ($N \gg 1$). He did this by taking the Fourier transform of the linearized equation of motion, finding the time dependence of the transform, and then reconstructing the behaviour of $\boldsymbol{u}(t)$ by taking the inverse transform subject to isotropic initial conditions and a suitable initial energy spectrum. He found that, for $\tau < t < l/u$, the turbulence kinetic energy decays as $(t/\tau)^{-1/2}$, while the turbulence becomes increasingly two-dimensional in the sense that the flow is independent of $\boldsymbol{x}_{\|}$. Of course, this is precisely what is seen in practice. Curiously, though, he observed an increase (rather than the expected decrease) in $\boldsymbol{u}_{\|}^2$. The kinetic energy is, in Moffatt's terminology, 'channelled' into the component parallel to $\boldsymbol{B}$. Any theory of MHD turbulence developed here should, in the limit of large N, reproduce $E \sim t^{-1/2}$, as well as the paradoxical growth in $\boldsymbol{u}_{\|}^2$. (In fact, we shall offer an explanation for the growth in $\boldsymbol{u}_{\|}^2$ based on our model problem analysed in §4.)

Sommeria & Moreau (1982) were, like Moffatt, concerned with MHD turbulence. They argued that the known tendency of turbulent structures to lengthen in the direction of $\boldsymbol{B}$ could be attributed to a diffusion-like process associated with (2.9). In particular, they suggested that provided $l_{\|} \gg l_{\perp}$ then (2.9) becomes

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} \sim \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} + \frac{l_{\perp}^2}{\tau}\frac{\partial^2\boldsymbol{\omega}}{\partial z^2}. \tag{3.1}$$

The implication is that vorticity tends to diffuse in a direction parallel to the magnetic field. Whether or not this diffusion produces any significant lengthing of $l_{\|}$ presumably depends on the size of N. When N is small, the vortex lines stretch and twist on a time scale of l/u, which is much smaller than τ, and it is difficult to infer much from (3.1). Conversely, when N is large the nonlinear terms vanish and (3.1) becomes a simple diffusion equation. We would then expect $l_{\|}$ to increase at a rate

$$l_{\|} \sim l_{\perp}(t/\tau)^{1/2}. \tag{3.2}$$

Expression (3.1) can be justified formally, for cases where $l_{\|} \gg l_{\perp}$, by taking a two-dimensional Fourier-transform of (2.9) in the (x, y)-plane. The diffusion coefficient then becomes $(k^2\tau)^{-1}$ where k is the wavenumber, $k_x^2 + k_y^2$. Note, however, that it is the Fourier-transform of $\boldsymbol{\omega}$ which diffuses, rather than $\boldsymbol{\omega}$ itself. Note also that there is no formal justification of (3.1) in cases where $l_{\|} \sim l_{\perp}$. Nevertheless, (3.1) provides a powerful way of visualizing the influence of $\boldsymbol{B}$ on the vorticity field. As noted by Sommeria & Moreau (1982), this pseudo-diffusion is the last vestige of Alfvén wave propagation in the limit of $Re_m \to 0$.

An alternative, global, view of magnetic damping has been provided by Davidson (1995). Unlike Moffatt or Sommeria & Moreau, Davidson was less concerned with turbulence than with the damping of large-scale motions such as isolated jets and vortices. The Reynolds number was taken to be high and so the inviscid equations of motion were used. However, as we shall see in §6, it is not difficult to extend the arguments to real viscous fluids. (Of course, viscous dissipation is an essential feature of any turbulent flow.)

Now, like Sommeria & Moreau (and indeed many others), Davidson noted that vorticity tends to propagate along the magnetic field lines. However, he attributed this to the need to conserve momentum in the face of a continual decline in kinetic energy. The argument may be established using linear momentum (in infinite domains) or angular momentum. We shall follow the latter line of argument.

Davidson starts with the observation that the Lorentz force cannot create or destroy the component of global angular momentum in the direction of $\boldsymbol{B}$. This follows from

$$(\boldsymbol{x}\times\boldsymbol{F})\cdot\boldsymbol{B} = \rho^{-1}[(\boldsymbol{x}\cdot\boldsymbol{B})\,\boldsymbol{J} - (\boldsymbol{x}\cdot\boldsymbol{J})\,\boldsymbol{B}]\cdot\boldsymbol{B} = -(\boldsymbol{B}^2/2\rho)\,\nabla\cdot[\boldsymbol{x}_{\perp}^2\,\boldsymbol{J}]$$

which integrates to zero over V.

Now in some geometries, such as flow in a sphere, the mechanical forces do not alter the global angular momentum of the fluid. Since the Lorentz force also leaves $\boldsymbol{H}_{\|}$ unaltered, we conclude that $\boldsymbol{H}_{\|}$ is conserved, despite the continual fall in energy demanded by (2.11). We might characterize such flows as conserving angular momentum in the face of continual Joule dissipation. This places a strong constraint on the way in which the flow may evolve. In particular, the magnetic damping cannot completely destroy the flow, but rather it must redistribute the angular momentum in such a way that E continually declines. The nature of this redistribution may be deduced from (2.4), which when combined with (2.11) yields the estimate

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim -\left(\frac{l_\perp}{l_\parallel}\right)^2 \frac{E}{\tau}. \tag{3.3}$$

Evidently, if the flow is to avoid destruction, as it must when $\boldsymbol{H}_\parallel$ is conserved, then the ratio $l_\parallel / l_\perp$ must continually increase. This argument is, of course, in accord with the predictions of Moffatt (1967) and Sommeria & Moreau (1982). However, unlike the local diffusion argument, or Moffatt's analysis, it is not restricted to large values of N. It is valid for any N, and so covers cases where the governing equations are nonlinear.

We conclude this section by noting that Davidson's analysis produces particularly simple results when applied to flow in a sphere. This is important as it provides the starting point for our analysis of MHD turbulence. We start by noting that conservation of $\boldsymbol{H}_\parallel$, in conjunction with the Schwarz inequality, places a lower bound on the kinetic energy of the flow:

$$E \geqslant H_\parallel^2 \Big/ \left[2 \int \boldsymbol{x}_\perp^2 \, \mathrm{d}V\right].$$

Thus any flow with non-zero $\boldsymbol{H}_\parallel$ must evolve to a steady state with finite kinetic energy. By virtue of (2.11) and (2.4) this final state must be strictly two-dimensional. Evidently, any angular momentum normal to the magnetic field is destroyed by the Lorentz force, leaving a flow consisting of one or more columnar eddies orientated parallel to the magnetic field. The rate of destruction of the transverse components of angular momentum is readily established. For a domain of arbitrary shape, the global magnetic torque is given by

$$\boldsymbol{T} = \frac{1}{2\rho} \int_V (\boldsymbol{x} \times \boldsymbol{J}) \, \mathrm{d}V \times \boldsymbol{B} = -\frac{\boldsymbol{H}_\perp}{4\tau} - \frac{\sigma}{2\rho} \left\{ \oint_S \Phi \boldsymbol{x} \times \mathrm{d}\boldsymbol{S} \right\} \times \boldsymbol{B}.$$

The first equality in the expression above is a standard result arising from $[2\boldsymbol{x} \times (\boldsymbol{J} \times \boldsymbol{B})]_i = [(\boldsymbol{x} \times \boldsymbol{J}) \times \boldsymbol{B}]_i + \boldsymbol{\nabla} \cdot (f_i \boldsymbol{J})$ where f_i is $[\boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{B})]_i$ (see, for example, Jackson 1962) while the second comes from substituting for $\boldsymbol{J}$ using (2.2) and then expanding the triple product $\boldsymbol{x} \times (\boldsymbol{u} \times \boldsymbol{B})$. For the particular case of a sphere, the surface integral vanishes and so the global angular momentum equation becomes

$$\frac{\partial \boldsymbol{H}}{\partial t} = -\frac{\boldsymbol{H}_\perp}{4\tau}$$

Equations (1.1) and (1.2) then follow:

$$\boldsymbol{H}_\parallel = \text{const.}, \quad \boldsymbol{H}_\perp(t) = \boldsymbol{H}_\perp(0) \exp[-4t/\tau].$$

As expected, $\boldsymbol{H}_\parallel$ is conserved, while $\boldsymbol{H}_\perp$ decays exponentially on a time scale of 4τ. (The physical explanation for the conservation of $\boldsymbol{H}_\parallel$ is given in Davidson 1995.)

The simplicity of this inviscid result is rather surprising, particularly as it applies for any value of N and so is valid when the stretching and twisting of vorticity is more vigorous than damping or pseudo-diffusion by $\boldsymbol{B}$. The nonlinearity of the local equations of motion has been circumvented through the use of global quantities and, in particular, the global angular momentum. One cannot help but notice the striking similarity between this result and the known tendency of a magnetic field to elongate turbulent eddies in the direction of $\boldsymbol{B}$, ultimately producing purely two-dimensional turbulence in the limit of large N. This 'model problem' is the starting point for our analysis of MHD turbulence. Of course, the first step is to introduce a small but finite viscosity. However, before doing this we shall look at the damping of inviscid isolated vortices, orientated either parallel or perpendicular to $\boldsymbol{B}$. We shall see that, as in the discussion above, global angular momentum holds the key to the behaviour to these flows. We start with transverse vortices.

4. Magnetic damping of transverse vortices

We now temporarily abandon flow in a sphere and consider the simplest possible configuration in which $\boldsymbol{B}$ is normal to the axis of rotation. Suppose our flow is strictly two-dimensional, confined to the (x, z)-plane, and bound by the cylindrical surface $x^2+z^2 = R^2$. We are interested particularly in isolated vortices whose characteristic radius, δ, is much less than R. We shall take the vortex to be initially axisymmetric and subject to a uniform magnetic, $\boldsymbol{B}$, imposed in the z-direction. Once again, we shall find that global angular momentum provides the key to determining evolution of the flow.

Since $\boldsymbol{B}$ and $\boldsymbol{\omega}$ are mutually perpendicular the electrostatic potential is zero, and so (2.5) gives the Lorentz force and magnetic torque as

$$\boldsymbol{F} = -(u_x/\tau)\,\hat{\boldsymbol{e}}_x, \tag{4.1}$$

$$T_y = -\tau^{-1}\int z u_x \,\mathrm{d}V = -H_y/2\tau. \tag{4.2}$$

Here H_y is the global angular momentum which may be expressed either in terms of $\boldsymbol{u}$ or else in terms of the two-dimensional streamfunction, ψ:

$$H_y = \int (z u_x - x u_z)\,\mathrm{d}V = 2\int z u_x \,dV = 2\int \psi \,\mathrm{d}V$$

(ψ is the y-component of the vector potential, a). It follows immediately that, even in the nonlinear regime, the angular momentum decays in a remarkably simple manner:

$$H_y(t) = H_y(0)\,\mathrm{e}^{-t/2\tau}. \tag{4.3}$$

Of course, this is the two-dimensional counterpart of (1.2). It is tempting to conclude, therefore, that the vortex decays on a time scale of 2τ. However, this appears to contradict (2.9) which, in the present context, simplifies to

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = -\frac{1}{\tau}\nabla^{-2}[\partial^2\omega/\partial z^2]. \tag{4.4}$$

(For simplicity we drop the subscript on $\boldsymbol{\omega}$.) In the spirit of Sommeria & Moreau (1982) we might write this in the form

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} \sim \frac{\delta^2}{\tau}\frac{\partial^2\omega}{\partial z^2}$$

and so we might anticipate (correctly) a continual diffusion of vorticity along the z-axis. In the limit of large N we have the simple diffusion equation,

$$\frac{\partial\omega}{\partial t} \sim \frac{\delta^2}{\tau}\frac{\partial^2\omega}{\partial z^2}$$

which suggest that the cross-section of the vortex distorts from a circle to a sheet on a time scale of τ. If this picture is correct, and we shall see that it is, this distortion should proceed in accordance with (3.2), and so we would expect l_z to increase as

$$l_z \sim \delta(t/\tau)^{1/2}. \tag{4.5}$$

This elongation of the eddy will cease only when the influence of the boundary is felt. We therefore have two conflicting views. On the one hand, (4.3) suggests that the flow is annihilated on a time scale of 2τ. On the other hand (4.5) suggests a continual evolution of the vortex until such time as the boundary plays an important role. This will occur when $l_z \sim R$, which requires a time of the order of $(R/\delta)^2 \tau$. We shall now show how these two viewpoints may be reconciled, and *en route* we shall shed some light on Moffatt's 'channelling' of kinetic energy into the z-component of motion.

Consider first the linear case where the magnetic field is relatively intense, so that $N \gg 1$. We further simplify the problem by insisting the boundaries are remote ($R \gg \delta$) so that we may consider flow in an infinite domain. This greatly simplifies the algebra, but at a cost. In order that all relevant integrals converge, particularly the angular momentum, we require that the integral of ψ converges and this limits our possible choice of initial conditions. (If the integral of ψ converges initially, then it subsequently converges for all time.) However, this sub-class of flows will suffice to show the general behaviour.

Consider the Fourier transform

$$\Psi(k_x, k_z) = 4 \int_0^\infty \int_0^\infty \psi(x, z) \cos(xk_x) \cos(zk_z) \, dx \, dz.$$

Then (4.4), in the form

$$\frac{\partial \psi}{\partial t} = -\frac{1}{\tau} \nabla^{-2} \frac{\partial^2 \psi}{\partial z^2} \tag{4.6}$$

requires that Ψ decays according to,

$$\Psi(k_x, k_z) = \Psi_0(k) \, e^{-(\cos^2\phi)\hat{t}}; \quad \cos\phi = k_z/k. \tag{4.7}$$

Here $\hat{t}$ is the dimensionless time t/τ, k is the magnitude of $\boldsymbol{k}$, and Ψ_0 is the transform of ψ at $t = 0$. The inverse transform then gives

$$\psi(\boldsymbol{x}, t) = \pi^{-2} \int_0^\infty \int_0^{\pi/2} e^{-(\cos^2\phi)\hat{t}} \cos(xk_x) \cos(zk_z) \, \Psi_0(k) \, k \, dk \, d\phi \tag{4.8}$$

which at large times ($t \gg \tau$) simplifies to

$$\psi(\boldsymbol{x}, t) = \frac{1}{2\pi(\pi\hat{t})^{1/2}} \int_0^\infty e^{-k^2 z^2/4\hat{t}} \cos(xk_x) \, \Psi_0(k) \, k \, dk. \tag{4.9}$$

(See Davidson 1995 for a discussion of the asymptotic behaviour of integrals of the form (4.8).) Evidently, for $t \gg \tau$, $\psi(\boldsymbol{x}, t)$ adopts the form

$$\psi(\boldsymbol{x}, t) \sim \hat{t}^{-1/2} F(z/\hat{t}^{1/2}, x), \tag{4.10}$$

where F is determined by the initial conditions. It would appear, therefore, that the arguments leading to (4.5) are substantially correct. An initially axisymmetric vortex progressively distorts into a sheet-like structure, with a longitudinal length scale given by (4.5). Note that (4.10) implies that $u_x \ll u_z$ while $u_z \sim \hat{t}^{-1/2}$. It follows that the kinetic energy of the eddy is progressively 'channelled' in the z-component of motion and that

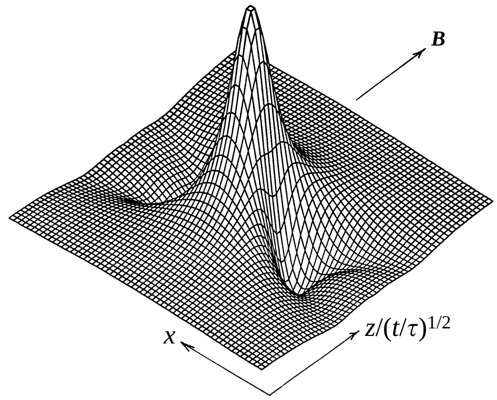

FIGURE 1. Magnetic damping of a transverse vortex at high N: the streamfunction at large times of an initially circular vortex. The z-axis is scaled by $(t/\tau)^{1/2}$, so that the vortex has, in fact, developed a sheet-like structure. Note the reverse eddies either side of the centreline.

the energy, E, declines as $E \sim (t/\tau)^{-1/2}$. Interestingly, both results are reminiscent of Moffatt's (1967) analysis of MHD turbulence.

Let us now consider a specific example. Suppose that the initial eddy structure is described by

$$\psi_0(r) = \Phi_0\, \mathrm{e}^{-r^2/\delta^2}; \quad r^2 = x^2 + z^2. \tag{4.11}$$

Then (4.9), which is valid at large times, may be integrated to give

$$\psi(\boldsymbol{x}, t) = \frac{\Phi_0}{(\pi \hat{t})^{1/2}} \frac{\zeta}{x^2} F(\zeta); \quad \zeta = \frac{x^2}{\delta^2 + z^2/\hat{t}} \tag{4.12}$$

where G is Kummer's hypergeometric function,

$$G(\zeta) = M(1, \tfrac{1}{2}, -\zeta).$$

Now expressions (4.10) and (4.12) seem to contradict (4.3), which predicts that the angular momentum decays as $\exp(-t/2\tau)$. However, (4.12) has an interesting property. For $t \gg \tau$, the global angular momentum, H_y, is

$$H_y = \frac{4\Phi_0 \delta^2}{\pi^{1/2}} \int_0^\infty (1 + x^2)^{-1/2} \int_0^\infty \zeta^{-1/2} G(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}x.$$

However, this integrates to zero since

$$\int_0^\infty \zeta^{-1/2} G(\zeta)\, \mathrm{d}\zeta = 0.$$

It would appear, then, that the structure of the flow at large times is such that the angular momentum is zero. The reason for this can be seen from figure 1 which shows the shape of ψ for $t \gg \tau$. Regions of reverse flow occur either side of the centreline of the vortex. This reverse flow has a magnitude which is just sufficient to cancel the angular momentum of the primary eddy.

These reverse eddies arise from a redistribution of momentum via the pressure forces. This may be seen from the linear momentum equation,

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{\nabla}\left(\frac{p}{\rho}\right) - \frac{u_x}{\tau} \hat{\boldsymbol{e}}_x.$$

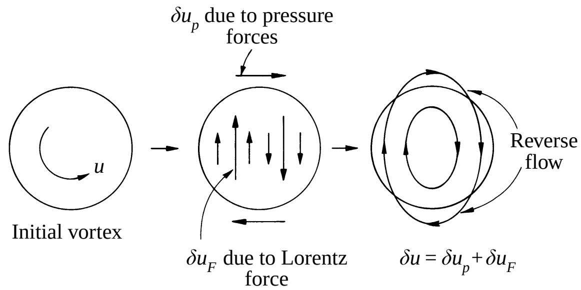

FIGURE 2. Magnetic damping of a transverse vortex at high N. The reverse flow arises from pressure forces acting outside the initial vortex.

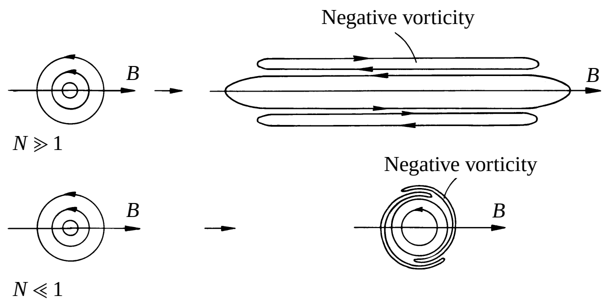

FIGURE 3. Magnetic damping of a transverse vortex at low and high N: a schematic of the evolution of an initially circular vortex. At low N the vortex remains nearly circular. At high N it develops a sheet-like structure.

In a time δt the velocity changes partially as a result of pressure forces and partially as a result of the Lorentz force. The latter contribution is

$$\delta \boldsymbol{u}_F = -\frac{\delta t}{\tau} u_x \hat{\boldsymbol{e}}_x.$$

Of course such a velocity increment contravenes continuity and it is the role of the pressure gradient to enforce conservation of volume. The difference between $\delta \boldsymbol{u}_F$ and the complete change in velocity, $\delta \boldsymbol{u}$, is shown schematically in figure 2. Clearly, it is the pressure force which induces the reverse flow above and below the symmetry axis.

We conclude, therefore, that the structure of the flow at large times is long and streaky, comprising vortex sheets of alternating sign. In short, the vorticity diffuses along the $\boldsymbol{B}$-lines in accordance with (4.5) while simultaneously adopting a layered structure which has zero net angular momentum, thus satisfying (4.3).

Let us now consider the other extreme of a weak magnetic field. When N is small or moderate the pseudo-diffusion of vorticity is much slower than advection. The problem is then a nonlinear one. We have been unable to find an exact solution of (4.4) in such cases. However, (4.3) still applies so that the angular momentum must disappear on a time scale of τ. It seems plausible that a structure such as that shown in figure 3 develops. That is, regions of negative vorticity grow, as in the large-N solution, but these are immediately swept around by the primary eddy to form a spiralled structure. Eventually, as the flow slows down, the value of N will rise and pseudo-diffusion will develop in accordance with (4.6).

In §§6 and 7 we shall see that this simple model problem provides one possible interpretation for the behaviour of MHD turbulence at high N. First, however, we look at isolated vortices where $\boldsymbol{B}$ is aligned with the axis of rotation. Again, angular momentum holds the key to the development of the flow. This time, however, the behaviour is quite different.

5. Magnetic damping of parallel vortices

We now examine the damping of an isolated vortex whose axis is aligned with $\boldsymbol{B}$. As before, we look separately at the high- and low-N limits. We are interested mainly in the general structure of these flows, in the anticipation that this will help us interpret our results for MHD turbulence. For simplicity, we restrict ourselves to axisymmetric vortices, described in terms of cylindrical polars (r, θ, z), with $\boldsymbol{B}$ parallel to z. As in §4, we neglect viscosity. Also, we shall assume that initial conditions are such that the integral of the angular momentum converges at $t = 0$. (If this integral converges at $t = 0$, then it converges for all $t > 0$.) Aspects of this problem have been touched upon by Davidson (1995).

Suppose we have an isolated region of intense swirl, of characteristic radius δ, in an otherwise quiescent liquid. We may uniquely define the instantaneous state of the flow using just two scalar functions: Γ, the angular momentum, and Ψ, the Stokes streamfunction. These are defined through the expressions

$$\boldsymbol{u} = \boldsymbol{u}_\theta + \boldsymbol{u}_p = (\Gamma/r)\,\hat{\boldsymbol{e}}_\theta + \boldsymbol{\nabla} \times [(\Psi/r)\,\hat{\boldsymbol{e}}_\theta], \tag{5.1}$$

$$\nabla_*^2 \Psi = \frac{\partial^2 \Psi}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right) = -r\omega_\theta. \tag{5.2}$$

Note that the velocity has been divided into azimuthal and poloidal components (subscripts θ and p). The Lorentz force, which is linear in $\boldsymbol{u}$, may be similarly divided, giving

$$\boldsymbol{F}_p = -\frac{u_r}{\tau}\hat{\boldsymbol{e}}_r = \frac{1}{r\tau}\frac{\partial \Psi}{\partial z}\hat{\boldsymbol{e}}_r, \tag{5.3}$$

$$F_\theta = -\frac{1}{\tau}\frac{J_r}{\sigma B} = \frac{1}{r\tau}\frac{\partial \phi}{\partial z}. \tag{5.4}$$

Here $\sigma B \phi$ is the Stokes streamfunction for $\boldsymbol{J}_p$, which, by virtue of Ohm's law, is related to Γ by

$$\nabla_*^2 \phi = -\partial \Gamma / \partial z. \tag{5.5}$$

We may now write down the governing equations for Γ and Ψ. These are the azimuthal components of the momentum and vorticity transport equations respectively:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \frac{1}{\tau}\frac{\partial \phi}{\partial z} = -\frac{1}{\tau}\frac{\partial^2}{\partial z^2}[\nabla_*^{-2}\Gamma], \tag{5.6}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\omega_\theta}{r}\right) = \frac{1}{r^4}\frac{\partial \Gamma^2}{\partial z} - \frac{1}{r^2\tau}\frac{\partial^2}{\partial z^2}[\nabla_*^{-2}(r\omega_\theta)]. \tag{5.7}$$

Note the appearance of the ubiquitous pseudo-diffusion terms. We might anticipate that angular momentum propagates along the magnetic field lines, and we shall see that this is substantially correct.

It is also useful to construct energy equations for the kinetic energies of the azimuthal and poloidal motions. These may be obtained by taking the product of $\boldsymbol{u}$

with the appropriate components of the momentum equation. It is not difficult to show (see Davidson 1995) that

$$\frac{\mathrm{d}E_\theta}{\mathrm{d}t} = -\int \frac{u_\theta^2}{r} u_r \,\mathrm{d}V - \frac{1}{\tau}\int (\nabla\phi)^2 \frac{\mathrm{d}V}{r^2}, \tag{5.8}$$

$$\frac{\mathrm{d}E_p}{\mathrm{d}t} = +\int \frac{u_\theta^2}{r} u_r \,\mathrm{d}V - \frac{1}{\tau}\int u_r^2 \,\mathrm{d}V. \tag{5.9}$$

The first term on the right of these equations represents the familiar exchange of energy between E_θ and E_p, reflecting the fact that a swirling vortex hoop can lower its azimuthal energy, E_θ, by centrifuging itself radially outward. This energy transfer underlies Rayleigh's centrifugal instability. The remaining two terms arise from the Lorentz force. They are, of course, negative and represent Joule dissipation. These energy equations are useful as they yield the following estimates:

$$\frac{\mathrm{d}E_\theta}{\mathrm{d}t} = \pm\frac{E_\theta}{t_p} - \frac{E_\theta}{\tau}\left(\frac{\delta}{l_z}\right)^2, \tag{5.10}$$

$$\frac{\mathrm{d}E_p}{\mathrm{d}t} = \mp\frac{E_\theta}{t_p} - \frac{E_p}{\tau}\left(\frac{\delta}{l_z}\right)^2, \tag{5.11}$$

where t_p is the turnover time of the poloidal motion, ω_θ^{-1}.

We shall now draw some general conclusions from (5.6)–(5.11). First, it is apparent from (5.6) that global angular momentum is conserved:

$$I_\Gamma = \int \Gamma \,\mathrm{d}V = \text{const.} \tag{5.12}$$

This is a special case of (1.1) and may be contrasted with the angular momentum of a transverse vortex. Secondly, for confined domains the energy of the flow has a lower bound. Specifically, the Schwarz inequality gives

$$E_\theta \geqslant I_\Gamma^2/2\int r^2 \,\mathrm{d}V. \tag{5.13}$$

Thirdly, as noted by Davidson (1995), any initial condition must evolve to a steady state of the form $(0, u_\theta(r), 0)$. This is true for any value of N, and requires only that the flow is confined. This result follows directly the energy equations (5.8) and (5.9), and from the energy bound (5.13). That is, we know that the flow eventually reaches a steady state with non-zero E_θ, at which time the Joule dissipation must vanish. Yet the dissipation disappears only when u_r and $\partial\Gamma/\partial z$ are both zero. This is a special case of the three-dimensional result of §3.

For infinite domains (5.13) does not apply. However, we can still use conservation of angular momentum to determine the manner in which the flow evolves. Combining (5.10) and (5.11) we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim -\left(\frac{\delta}{l_z}\right)^2 \frac{E}{\tau}.$$

Thus the total energy declines as

$$E \sim E_0 \exp\left[-\int_0^t (\delta/l_z)^2 \,\mathrm{d}\hat{t}\right].$$

If angular momentum is to be conserved then there are only two ways in which this decline in energy can be accommodated. Either l_z increases with time to reduce the dissipation, thus avoiding the exponential decline in energy, or else the angular momentum centrifuges itself radially outward, allowing the energy to decline despite the conservation of I_Γ. We shall see that axial spreading of angular momentum is typical of high-N flows, while the radial spreading of angular momentum is characteristic of low-N flows.

Let us now consider separately the limits of high and low N. When N is large, the azimuthal and poloidal motions are virtually decoupled. This follows from (5.10) and (5.11). Specifically, N is of the order of t_p/τ, so that when N is large the energy exchange terms are negligible by comparison with the Joule dissipation. If E_p is initially small (of the order of $N^{-1}E_\theta$), it remains small. The flow is then governed by the simple linear equation

$$\frac{\partial \Gamma}{\partial t} = -\frac{1}{\tau}\frac{\partial^2}{\partial z^2}[\nabla_*^{-2}\Gamma]. \tag{5.14}$$

We expect, therefore, that any localized region of swirl will diffuse along the magnetic field lines, at a rate determined by

$$l_z \sim \delta(t/\tau)^{1/2}. \tag{5.15}$$

As in the previous section, we may confirm this by taking the Fourier transform of (5.14). Suppose that the flow is unbounded and let U be the first-order Hankel-cosine transform of u_θ:

$$U(k_r, k_z) = 4\pi \int_0^\infty \int_0^\infty \Gamma(r,z)\, J_1(k_r r)\cos(k_z z)\,\mathrm{d}r\,\mathrm{d}z.$$

Then (5.14) shows that U decays as

$$U = U_0\,\mathrm{e}^{-(\cos^2\phi)\hat{t}}, \quad \cos\phi = k_z/k.$$

As before, $\hat{t}$ is the dimensionless time t/τ, U_0 represents the initial condition, and k is the magnitude of $\boldsymbol{k}$. We can now determine Γ by taking the inverse transform, which yields

$$\Gamma = \frac{r}{2\pi^2}\int_0^\infty \int_0^\infty U_0(k_r,k_z)\,\mathrm{e}^{-(\cos^2\phi)\hat{t}} J_1(k_r r)\cos(k_z z)\,k_r\,\mathrm{d}k_r\,\mathrm{d}k_z. \tag{5.16}$$

We are interested particularly in the asymptotic form of Γ at large times. To that end it is convenient to introduce a new variable

$$q = (\tfrac{1}{2}\pi - \phi)\,t^{1/2}.$$

If we now consider the limit of large $\hat{t}$, then (5.16) simplifies to

$$\Gamma(\hat{t}\to\infty) = \frac{r\hat{t}^{-1/2}}{2\pi^2}\int_0^\infty \int_0^\infty [kU_0(k,k_z)]\,\mathrm{e}^{-q^2} J_1(kr)\cos(kqz/\hat{t}^{1/2})\,k\,\mathrm{d}k\,\mathrm{d}q. \tag{5.17}$$

This confirms that, at large times, the distribution of angular momentum is of the form

$$\Gamma(\boldsymbol{x},t) = (t/\tau)^{-1/2} F(r, z/(t/\tau)^{1/2}). \tag{5.18}$$

Note the similarity between (5.18) and the evolution of ψ for two-dimensional transverse vortices. As expected, the angular momentum propagates along the z-axis at a rate governed by (5.15), but decays according to $u_\theta \sim (t/\tau)^{-1/2}$. The energy of the vortex therefore declines at a rate

$$E \sim (t/\tau)^{-1/2}$$

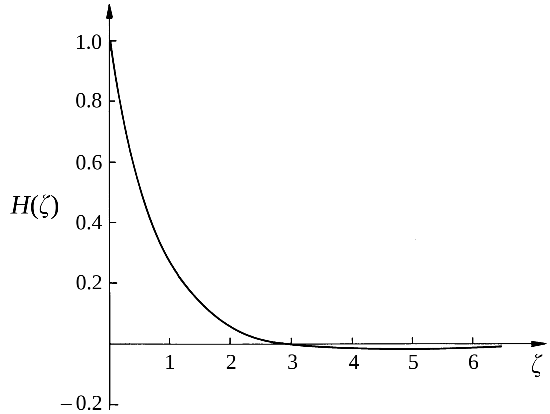

FIGURE 4. Magnetic damping of a parallel vortex at high N: $H(\zeta)$, the distribution of swirl with radius at large times (see equation (5.19)). Note the reverse rotation at large radii.

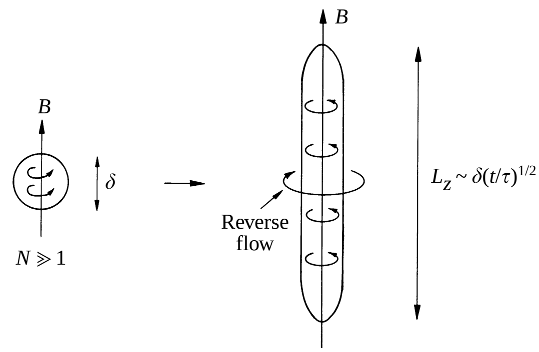

FIGURE 5. Magnetic damping of a parallel vortex at high N. The figure shows schematically the structure of the flow at large times.

which is exactly the same as for the transverse vortex of §4.

By way of an example suppose that, at $t = 0$, we have a spherical blob of swirling fluid, so that our initial condition is

$$\Gamma_0(r, z) = \Omega r^2 \exp[-(r^2+z^2)/\delta^2].$$

This transforms to

$$U_0 = \tfrac{1}{2}\Omega \pi^{3/2} \delta^5 k_r \exp[-\tfrac{1}{4}\delta^2 k^2]$$

and so (5.17) may be integrated to give

$$u_\theta(\hat{t} \to \infty) = \Omega \delta \hat{t}^{-1/2} \tfrac{3}{4} \pi^{1/2} \left(\frac{\delta}{r}\right)^4 \zeta^{5/2} H(\zeta), \quad \zeta = \frac{r^2}{\delta^2 + z^2/\hat{t}}. \tag{5.19}$$

here $H(\zeta)$ is the hypergeometric function

$$H(\zeta) = M(\tfrac{5}{2}, 2, -\zeta).$$

The shape of $H(\zeta)$ is shown in figure 4. Curiously, at large ζ the function H becomes negative ($H \sim -\zeta^{-5/2}/2\pi^{1/2}$), so that the primary vortex is surrounded by a region of counter-rotating fluid. This was predicted in Davidson (1995) and may be attributed to the way in which the induced currents recirculate back through quiescent regions outside the initial vortex. We conclude, therefore, that the asymptotic structure of a vortex aligned with $\boldsymbol{B}$ is as shown schematically in figure 5. It is long and elongated,

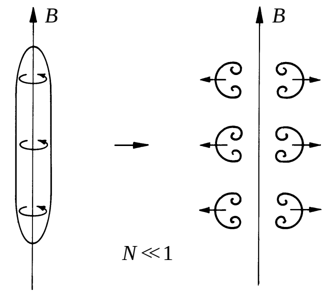

FIGURE 6. Magnetic damping of a parallel vortex at low N. The vortex will disintegrate through hoops of swirling fluid centrifuging themselves radially outward.

cigar-like in shape, and quite different in structure to the transverse vortex shown in figure 3. Curiously, though, despite the fact that the two classes of vortices adopt very different structures, their energies both decay as $(t/\tau)^{-1/2}$.

We now turn our attention to the case where N is low. Our energy equations now suggest that Joule dissipation is negligible on time scales of the order of t_p, so the flow evolves in accordance with the undamped Euler equations. Our initial blob of swirling fluid, which is centrifugally unstable, will centrifuge itself radially outward. The experiments of the Japan Society of Mechanical Engineers (1988), and the computations of Davidson (1993), suggest that this occurs through the angular momentum organizing itself into one or more ring-shaped vortices. These propagate radially outward with the characteristic mushroom-like structure of a thermal plume. This is shown schematically in figure 6.

The formation time for these plume-like structures is of the order of the turnover time of the original vortex. The picture which emerges at low N, therefore, in one in which the vortex breaks up on a time scale of one turnover time, and in a manner quite unaffected by the magnetic field. The role of $\boldsymbol{B}$ is merely to provide relatively weak dissipation which reduces the energy of the flow by an amount $\sim NE_\theta$ during the lifetime of the original vortex.

We now turn our attention to MHD turbulence.

6. The role of angular momentum in MHD turbulence

We have already seen, in §§1 and 2, that the global angular momentum of fluid in a uniform magnetic field behaves in a particularly simple way. We shall use this to estimate the rate of decay of energy and the rate of growth of anisotropy in MHD turbulence. However, before looking at the implications of (1.1) and (1.2), it is useful to consider first the role of angular momentum in the decay of conventional, isotropic turbulence. This relationship was first pointed out by Landau & Lifshitz (1959) in a reinterpretation of Kolmogorov's $t^{-10/7}$ decay law. We start with Kolmogorov's derivation. Our review is brief, but a more detailed discussion may be found in Monin & Yaglom (1975), Hinze (1975), or Lesieur (1987). Kolmogorov's starting point was Loitsyansky's integral.

Loitsyansky asserted (incorrectly as it turns out) that an integral of the longitudinal velocity correlation function Q_{ll}

$$L = \int_0^\infty r^4 Q_{ll}\,\mathrm{d}r \qquad (6.1)$$

is conserved in isotropic turbulent flow. If v is a typical velocity of the large scales, and l the integral length scale, this gives

$$v^2 l^5 = \text{const.} \tag{6.2}$$

Kolmogorov took advantage of this result. In addition, he invoked the concept of the energy cascade, according to which the energy dissipation rate per unit mass is $\sim v^3/l$, so that

$$-\frac{\mathrm{d}v^2}{\mathrm{d}t} \sim v^3/l. \tag{6.3}$$

Combining these results yields an energy decay rate of

$$v^2 \sim t^{-10/7}. \tag{6.4}$$

This power law is, in fact, reasonably in line with the experimental evidence. Now the invariance of Loitsyansky's integral, L, was asserted on the basis of a dynamic equation for the velocity correlation function (the Kármán–Howarth equation) which is perfectly rigorous. However, it also relies on the assumption that velocity correlations decay rapidly with distance, so that certain surface integrals could be neglected (see, for example, Hinze 1975). We shall return to this point shortly. In the meantime, we note that Landau & Lifshitz (1959) gave a simpler physical explanation for the conservation of L, based on the conservation of angular momentum. (This was later extended by Lumley 1966). Their argument went as follows.

Suppose an isotropic turbulent flow is contained in a sphere whose radius, R, is very much larger than the integral scale of the turbulence, l. The net angular momentum of the fluid need not be conserved because of shear stresses at the boundary. However, as the radius of sphere increases, this surface effect becomes less important and ultimately, for $R \gg l$, the surface stresses will not influence the motion on the time scale of its decay. In this sense, then, angular momentum is conserved. Consider now one component of angular momentum, say H_z:

$$H_z = \int (xu_y - yu_x)\,\mathrm{d}V = 2\int xu_y\,\mathrm{d}V = -2\int yu_x\,\mathrm{d}V.$$

We might write the square of H_z as

$$H_z^2 = 4\int_V xu_y\,\mathrm{d}V \int_{V'} x'u'_y\,\mathrm{d}V' = 4\int_V\int_{V'} xx'u_y u'_y\,\mathrm{d}V'\,\mathrm{d}V.$$

Since the integral of u_y is zero we can rewrite this in the form

$$H_z^2 = -2\iint (x-x')^2 u_y u'_y\,\mathrm{d}V'\,\mathrm{d}V.$$

However, we could equally have expressed H_z in terms of yu_x, and so it follows that

$$H_z^2 = -\iint [(x-x')^2 u_y u'_y + (y-y')^2 u_x u'_x]\,\mathrm{d}V'\,\mathrm{d}V. \tag{6.5}$$

Adding all three components of $\boldsymbol{H}^2$, and noting that terms of the type $xx'u_x u'_x$ integrate to zero, we obtain

$$\boldsymbol{H}^2 = -\iint [(\boldsymbol{x}-\boldsymbol{x}')^2 \boldsymbol{u}\cdot\boldsymbol{u}']\,\mathrm{d}V'\,\mathrm{d}V.$$

Finally, we ensemble average $\boldsymbol{u}\cdot\boldsymbol{u}'$ for each pair of points separated by a fixed distance, $\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{x}'$, to give

$$\boldsymbol{H}^2 = -\int\left[\int r^2\overline{\boldsymbol{u}\cdot\boldsymbol{u}'}\,\mathrm{d}\boldsymbol{r}\right]\mathrm{d}\boldsymbol{x}. \tag{6.6}$$

The implication is that

$$I_{AV} = \left[-\int r^2\overline{\boldsymbol{u}\cdot\boldsymbol{u}'}\,\mathrm{d}\boldsymbol{r}\right]_{AV} = \boldsymbol{H}^2/V \tag{6.7}$$

is an invariant of the motion. (The subscript AV indicates a spatial average over V.) The final step in Landau & Lifshitz's argument was to assume that the velocity correlation $\overline{\boldsymbol{u}\cdot\boldsymbol{u}'}(\boldsymbol{r})$ decays rapidly with $|\boldsymbol{r}|$, so that far-field contributions to the integral

$$\int r^2\overline{\boldsymbol{u}\cdot\boldsymbol{u}'}\,\mathrm{d}\boldsymbol{r}$$

are small. In such a situation only those velocity correlations taken close to the boundary are aware of the presence of this surface, and in this sense the turbulence is approximately homogeneous. To leading order in l/R we then have

$$\boldsymbol{H}^2 = -V\int r^2\overline{\boldsymbol{u}\cdot\boldsymbol{u}}\,\mathrm{d}\boldsymbol{r} = VI. \tag{6.8}$$

For the particular case of isotropic turbulence, $I = 8\pi L$ (see Hinze 1975) and so conservation of angular momentum implies conservation of Loitsyansky's integral and the $t^{-10/7}$ decay law follows.

The problem with these arguments is, of course, that the velocity correlation

$$Q_{ij}(\boldsymbol{r}) = \overline{u_i(\boldsymbol{x})\,u_j(\boldsymbol{x}+\boldsymbol{r})}$$

need not decay that rapidly with distance. (Typically it decays at r^{-5}.) There are three consequences of this. First, the integral I (for homogeneous turbulence) or L (for homogeneous isotropic turbulence) need not converge. Secondly, the last step in Landau & Lifshitz's argument cannot be justified. That is, while conservation of angular momentum implies that I_{AV} is an invariant ((6.7) makes no assumption about Q_{ij}), we cannot infer that I is a dynamical invariant. Thirdly, Loitsyansky's proof that L is an invariant is flawed. In fact, it is well known that Loitsyansky's integral is, in general, time-dependent. These last two points are closely related. (To avoid the difficulty associated with the boundary $r = R$ in Landau & Lifshitz's derivation of (6.8), Monin & Yaglom (1975) suggest a procedure in which the local angular momentum density is multiplied by an exponential decay factor $\exp(-\alpha r^2)$ before integration. The limit of $\alpha \to 0$ is then taken. However, this leads to the same conclusion as Landau & Lifshitz's argument.)

The reason why Q_{ij} decays algebraically with r rather than exponentially, as was assumed in early theories of turbulence, is related to the incompressibility of the fluid. The pressure forces, which enforce conservation of volume, can instantaneously propagate information to all points within the medium. A fluctuation in velocity at one location is immediately felt everywhere in the fluid due to the pressure forces. These, in turn, induce accelerations in the far field which give rise to long-range velocity correlations. This was investigated by Batchelor & Proudman (1956).

As noted in Hinze (1975), the spatial rate of decay of Q_{ij} depends partially on whether the turbulence is isotropic and partially on the form of the energy spectrum,

$E(k)$, at small k. When $E(k) \sim k^4$, which is usually taken to be the case for homogeneous turbulence, Q_{ij} decays as r^{-5} (Batchelor & Proudman 1956). This is sufficiently rapid for moments of the form

$$I_{ijmn} = \int Q_{ij} r_n r_m \,\mathrm{d}\boldsymbol{r}$$

to converge (so that I and L converge) but not fast enough for I, L or I_{ijmn} to be invariants of the motion. (I_{ijmn} is a dynamical invariant when Q_{ij} decays exponentially with r.) Here the convergence of I_{ijmn} is a consequence of symmetry, whereby terms of order r^{-5} integrate to zero. When the turbulence is isotropic, and $E \sim k^4$, Q_{ij} decays as r^{-6}, or faster. Again, this ensures that I, L and I_{ijmn} converge but, as with the anisotropic case, it does not ensure that they are invariants of the flow.

If $E(k) \sim k^2$, on the other hand, Q_{ij} decays as r^{-3}. Such a spectrum has been suggested by Saffman (1967) who noted that I and L diverge for such a flow, but that one could establish a new class of invariants related to linear (rather than angular) momentum. Indeed, such a spectrum seems to rely on the global linear momentum being non-zero, which raises interesting questions about its applicability to flow in finite domains. We shall return to this case later. In the meantime, however, we shall restrict ourselves to $E \sim k^4$ spectra.

Now it turns out that, despite the variation of L due to long-range velocity correlations, Kolmogorov's $t^{-10/7}$ law is not too far out of line with the experimental data. Experiments suggest that the energy decays as $t^{-1.25}$–$t^{-1.36}$ (Lesieur 1987). As Hinze (1975) and Lesieur (1987) note, this is because Loitsyansky's integral changes only very slowly with time, reflecting the rather weak influence of the long-range velocity correlations. The best current estimates of the decay law, which rest on the use of intricate phenomenological closure schemes, suggest a $t^{-1.38}$ decay (Lesieur 1987), which is not that different from Kolmogorov's $t^{-1.43}$. As Lesieur notes 'an assumption like the invariance with time of L would not greatly alter the decay of turbulence'.

We shall now examine the implications of (1.1) and (1.2) for MHD turbulence. We shall assume an energy spectrum of the form $E \sim k^4$ at small k, and follow an argument analogous to that of Landau & Lifshitz (1959). Initially, we shall make no assumption about the long-range behaviour of Q_{ij}. Subsequently, however, we shall tentatively explore the consequences of assuming that the long-range velocity correlations are weak, as seems to be the case in conventional isotropic turbulence. Of course, in MHD turbulence, we have to contend with the additional long-range forces which arise from $\boldsymbol{B}$. However, we argue in §7 that in axisymmetric turbulence these additional forces are no more dangerous than the pressure forces. For example, as in conventional homogeneous turbulence, $Q_{ij}(\boldsymbol{r})$ decays as r^{-5}, and integrals such as I converge because terms in r^{-5} cancel. (Unlike anisotropic turbulence in rotating or stratified media, velocity fluctuations in MHD turbulence are not propagated by internal waves, but rather by pseudo-diffusion, which is less efficient at dispatching energy to the far field.)

Our starting point is the same as that of Landau & Lifshitz (1959). Suppose that our fluid is held in a sphere whose radius is much larger than the integral scale of the turbulence. We assume that the turbulence is initially isotropic and that the Reynolds number, Re, is high. We expect the parallel integral scale so increase with time, and so we limit our time (or choose our sphere radius) to be such that $(l_{\parallel})_{max} \ll R$. We are then free to ignore surface effects, such as drag due to shear stresses, Ekman pumping, or concentrated dissipation in Hartmann layers. (Note that the restriction $(l_{\parallel})_{max} \ll R$ means that we cannot capture the final stages of decay when $\boldsymbol{B}$ is large, i.e. situations where the turbulence becomes pseudo-two-dimensional.)

It is convenient to define the interaction parameter in terms of the perpendicular integral scale,

$$N(t) = (l_{\perp}/u_{\perp})/\tau = \tau_{t0}/\tau.$$

We shall place no limit on N, other than $N \ll Re$. Of course, as the turbulence decays, N will increase along with the eddy turnover time, τ_{t0}. Curiously, though, this does not necessarily imply that the Lorentz forces become relatively more important. From (2.8) we have,

$$\boldsymbol{F}/(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) \sim N(l_{\perp}/l_{\parallel})^2.$$

Evidently, the relative size of the Lorentz and inertial forces depends not only on the instantaneous value of N, but also on the degree of anisotropy. Consider, for example, the case where $N_0 \gg 1$. Initially, the Lorentz forces are dominant and the problem is a linear one. We might expect that both parallel and perpendicular eddies will evolve in a manner similar to the isolated vortices discussed in §§4 and 5. In such a case the energy of the flow decays as $E \sim (t/\tau)^{-1/2}$ while anisotropy develops according to $(l_{\parallel}/l_{\perp}) \sim (t/\tau)^{1/2}$. It follows that

$$N(t) \sim N_0(t/\tau)^{1/4},$$

$$\boldsymbol{F}/(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) \sim N_0(t/\tau)^{-3/4}.$$

Thus, despite the fact that N increases as $t^{1/4}$, the Lorentz forces become weaker with time and eventually nonlinear behaviour will set in (Moreau 1990).

It should also be borne in mind that the smallest scale of the turbulence, the Kolmogorov microscale, has a turnover time much less than τ_{t0}, of $Re^{-1/2}\tau_{t0}$. Since we are assuming that $N \ll Re$, as is invariably the case in practice, the Lorentz forces act only on the largest scales of the turbulence. Two quite different pictures then emerge, depending on whether N is large or small. When N is large the primary source of dissipation is the Joule effect, and this operates only on the large scales. As the motion decays, the energy spectrum $E(k)$ 'collapses' from the low-k end. The smaller scales are only aware of the magnetic damping to the extent that their supply of energy via the energy cascade is diminished. When N is small, on the other hand, the primary source of dissipation is viscous shear acting at the smallest scales. Here the energy spectrum collapses from the high-k end, with the largest scales being the last to decay.

With this background in mind we now turn to the main result of this section. Our starting point is (6.5) and (6.6), applied this time to MHD turbulence. For the parallel and perpendicular components of $\boldsymbol{H}$ we have

$$\boldsymbol{H}_{\parallel}^2 = -\int\left[\int \boldsymbol{r}_{\perp}^2\,\overline{\boldsymbol{u}_{\perp}\cdot\boldsymbol{u}'_{\perp}}\,\mathrm{d}\boldsymbol{r}\right]\mathrm{d}\boldsymbol{x}, \tag{6.9}$$

$$\boldsymbol{H}_{\perp}^2 = -2\int\left[\int \boldsymbol{r}_{\parallel}^2\,\overline{\boldsymbol{u}_{\perp}\cdot\boldsymbol{u}'_{\perp}}\,\mathrm{d}\boldsymbol{r}\right]\mathrm{d}\boldsymbol{x} \tag{6.10}$$

and

$$\boldsymbol{H}_{\perp}^2 = -2\int\left[\int \boldsymbol{r}_{\perp}^2\,\overline{\boldsymbol{u}_{\parallel}\cdot\boldsymbol{u}'_{\parallel}}\,\mathrm{d}\boldsymbol{r}\right]\mathrm{d}\boldsymbol{x}. \tag{6.11}$$

Combining this with (1.1) and (1.2) we obtain the key result of this section:

$$[I_1]_{AV} = \left[-\int \boldsymbol{r}_{\perp}^2\,\overline{\boldsymbol{u}_{\perp}\cdot\boldsymbol{u}'_{\perp}}\,\mathrm{d}\boldsymbol{r}\right]_{AV} = \text{const.}, \tag{6.12}$$

$$[I_2]_{AV} = \left[-\int \boldsymbol{r}_{\parallel}^2\,\overline{\boldsymbol{u}_{\perp}\cdot\boldsymbol{u}'_{\perp}}\,\mathrm{d}\boldsymbol{r}\right]_{AV} = [I_2]_0\exp[-t/2\tau], \tag{6.13}$$

$$[I_3]_{AV} = \left[-\int \boldsymbol{r}_{\perp}^2\,\overline{\boldsymbol{u}_{\parallel}\cdot\boldsymbol{u}'_{\parallel}}\,\mathrm{d}\boldsymbol{r}\right]_{AV} = [I_3]_0\exp[-t/2\tau]. \tag{6.14}$$

It should be emphasized that these are fully nonlinear results, applicable to any value of N. So far we have made no assumption about the decay of Q_{ij} at large r. If (but only if) the long-range velocity correlations are weak, so that the inhomogeneous region near the surface is small, we have

$$I_1(t) = -\int \boldsymbol{r}_\perp^2\, \overline{\boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp}\, \mathrm{d}\boldsymbol{r} \approx I_1(0), \tag{6.15}$$

$$I_2(t) = -\int \boldsymbol{r}_\parallel^2\, \overline{\boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp}\, \mathrm{d}\boldsymbol{r} \approx I_2(0) \exp[-t/2\tau], \tag{6.16}$$

$$I_3(t) = -\int \boldsymbol{r}_\perp^2\, \overline{\boldsymbol{u}_\parallel \cdot \boldsymbol{u}'_\parallel}\, \mathrm{d}\boldsymbol{r} \approx I_3(0) \exp[-t/2\tau]. \tag{6.17}$$

In such cases the behaviour of the global angular momentum gives a direct indication of the rate of growth of anisotropy as measured by the moments of the velocity correlations. If (6.15)–(6.17) are valid, they represent much stronger statements than (6.12)–(6.14).

Now expressions (6.15)–(6.17) may be derived in a mathematically more formal (if physically less revealing) manner. The method is essentially the same as that originally used to establish the invariance of Loitsyansky's integral for cases where the far-field velocity correlations are very weak. The approach is to assume from the outset that the velocity correlations decay rapidly with distance. For completeness we briefly describe this method. It should be noted, however, that such a derivation is not rigorous because in practice the velocity correlations could decay rather slowly with distance (i.e. as r^{-5}). This second derivation has the advantage, though, of showing exactly where the assumption of small Q_{ij} at large r enters into the analysis.

The starting point is to derive a dynamic equation for the correlation of the angular momentum at two locations A and B. Let $\boldsymbol{h}$ denote the angular momentum $\boldsymbol{x} \times \boldsymbol{u}$. The equation of motion for h_i at the point A is

$$\{\partial h_i/\partial t + \boldsymbol{u} \cdot \nabla h_i = [\nabla \times (p\boldsymbol{x}/\rho)]_i + (\boldsymbol{x} \times \boldsymbol{F})_i\}_A.$$

Multiplying this by h_j at location B gives

$$h_{jB}(\partial h_{iA}/\partial t) = -\boldsymbol{u}_A \cdot \nabla_A (h_{iA} h_{jB}) + [\nabla_A \times (h_{jB} p\boldsymbol{x}/\rho)]_i + h_{jB}(\boldsymbol{x} \times \boldsymbol{F})_{iA}.$$

Here we note that h_{jB} may be treated as a constant in a differentiation at point A. In the same way we may multiply the equation of motion for h_{jB} by h_{iA}. Combining the two results and then averaging gives the required correlation equation:

$$\frac{\partial}{\partial t}(\overline{h_{iA} h_{jB}}) = f_{ij}(\boldsymbol{u}, p) + \overline{h_{jB}(\boldsymbol{x} \times \boldsymbol{F})_{iA}} + \overline{h_{iA}(\boldsymbol{x} \times \boldsymbol{F})_{jB}}.$$

Here $f_{ij}(\boldsymbol{u}, p)$ denotes symbolically all of the conventional inertial and pressure terms which would appear on the right-hand side of such an equation in the absence of a magnetic field. Now $\overline{h_{iA} h_{jB}}$ is a function of time and of $\zeta = \boldsymbol{x}_B - \boldsymbol{x}_A$. Integrating with respect to ζ we have

$$\frac{\partial}{\partial t}\int (\overline{h_{iA} h_{jB}})\, \mathrm{d}\zeta = \int f_{ij}(\boldsymbol{u}, p)\, \mathrm{d}\zeta + \int [\overline{h_{jB}(\boldsymbol{x} \times \boldsymbol{F})_{iA}} + \overline{h_{iA}(\boldsymbol{x} \times \boldsymbol{F})_{jB}}]\, \mathrm{d}\zeta.$$

However, the first integral on the right vanishes for rapidly decaying velocity correlations. We know this since I_{ijmn} is an invariant of conventional turbulence under such conditions. We then have

$$\frac{\partial}{\partial t}\int(\overline{h_{iA}\,h_{jB}})\,\mathrm{d}\zeta = \int[\overline{h_{jB}(\boldsymbol{x}\times\boldsymbol{F})_{iA}}+\overline{h_{iA}(\boldsymbol{x}\times\boldsymbol{F})_{jB}}]\,\mathrm{d}\zeta. \tag{6.18}$$

The remaining integral on the right may be simplified if we note that the magnetic torque per unit volume may be written in the form

$$2\rho(\boldsymbol{x}\times\boldsymbol{F})_i = \boldsymbol{\nabla}\cdot[\boldsymbol{J}(\boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{B}))_i]+\sigma[\boldsymbol{\nabla}\times(\Phi\boldsymbol{x})\times\boldsymbol{B}]_i + \tfrac{1}{2}\sigma\boldsymbol{\nabla}\cdot[(\boldsymbol{x}\times\boldsymbol{B})_i\,(\boldsymbol{B}\cdot\boldsymbol{x})\,\boldsymbol{u}]-\rho(\boldsymbol{x}\times\boldsymbol{u})_\perp/2\tau.$$

Now the first three terms on the right-hand side lead to surface integrals in (6.18). The term involving Φ integrates to zero if we take the surface to be a sphere of large radius, while the first and third terms lead to surface integrals involving the far-field correlations $\overline{J_i u_j}$ and $\overline{u_i u_j}$. Since all correlations are assumed to decay rapidly, the surface integrals vanish. We are left with

$$\frac{\partial}{\partial t}\int(\overline{h_{iA}\,h_{jB}})\,\mathrm{d}\zeta = -(4\tau)^{-1}\int(\overline{h_{jB}\,h_{iA\perp}}+\overline{h_{iA}\,h_{jB\perp}})\,\mathrm{d}\zeta.$$

In particular, this gives equations for $\boldsymbol{h}_\perp$ and $\boldsymbol{h}_\parallel$:

$$\frac{\partial}{\partial t}\int(\overline{h_{\parallel A}\,h_{\parallel B}})\,\mathrm{d}\zeta = 0, \tag{6.19}$$

$$\frac{\partial}{\partial t}\int(\overline{\boldsymbol{h}_{\perp A}\cdot\boldsymbol{h}_{\perp B}})\,\mathrm{d}\zeta = -(2\tau)^{-1}\int(\overline{\boldsymbol{h}_{\perp A}\cdot\boldsymbol{h}_{\perp B}})\,\mathrm{d}\zeta. \tag{6.20}$$

Expressions (6.19) and (6.20) lead directly to (6.15)–(6.17). Of course, a derivation of the type given above is flawed because we have no right to assume that the velocity conditions decay rapidly with distance. However, this second argument is useful in as much as it does show explicitly the way in which estimates (6.15)–(6.17) depend on the weakness of the long-range correlations.

In §8 we shall give a physical interpretation of (6.12)–(6.17) based on our study of isolated vortices in §§4 and 5. We also explore the consequences of these relationships for the rate of decay of energy and the rate of growth of anisotropy. First, however, we examine the role of the Lorentz forces in establishing long-range velocity correlations. We have in mind two particular points. First, we wish to establish whether the integrals I_1, I_2 and I_3 converge. Secondly, we would like to know if the long-range correlations associated with the magnetic field are in some way more important than those which appear in conventional turbulence. We shall see that, provided the turbulence remains axisymmetric, the far-field influence of the Lorentz forces is no stronger than that of pressure. We show that I_1, I_2 and I_3 are likely to be convergent, and that Q_{ij} decays as r^{-5}, just as in conventional, homogeneous turbulence. (The convergence properties of these integrals could in principle depend on the manner in which the domain is extended to infinity. Here we first integrate over a sphere of finite radius and then let that radius tend to infinity.)

7. Long-range velocity correlations induced by the Lorentz forces

We wish to determine the influence of the magnetic field on the long-range velocity correlations in axisymmetric turbulence. Here we adopt the same approach as that used to establish the influence of pressure forces on the velocity correlation in conventional homogeneous turbulence (Batchelor & Proudman 1956). We assume that, at some initial instant, the turbulence is axisymmetric and homogeneous and that all velocity correlations decay exponentially with r. (This latter condition is related to the manner in which the turbulence is set up.) We then use our equations of motion to determine the rate of change of Q_{ij} at $t = 0$. This tells us the structure of Q_{ij} immediately after $t = 0$, and we take this structure to be representative of Q_{ij} at all times.

Consider the Kármán–Howarth equation for homogeneous, but not necessarily isotropic, turbulence. This is an equation of motion for Q_{ij} obtained by applying the Navier–Stokes equation to two points, A and B. For MHD turbulence we must add the terms arising from the Lorentz force. It is not difficult to show that, if $\zeta = \boldsymbol{x}_B - \boldsymbol{x}_A$, then

$$\frac{\partial}{\partial t} Q_{ij}(\zeta) = \frac{1}{2\pi\tau}\frac{\partial^2}{\partial \zeta_z^2}\left[\int \frac{Q_{ij}(\zeta')}{|\zeta-\zeta'|}\,\mathrm{d}\zeta'\right] + (\text{inertial and pressure terms}).$$

We now replace ζ by $\boldsymbol{x}$ and use $T_{ij}(\boldsymbol{u}, p)$ to denote the conventional presure and inertial terms:

$$\frac{\partial Q_{ij}}{\partial t} = T_{ij}(\boldsymbol{u}, p) + \frac{1}{2\pi\tau}\frac{\partial^2}{\partial z^2}\left[\int \frac{Q'_{ij}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}\boldsymbol{x}'\right]. \tag{7.1}$$

We are particularly interested in large values of $\boldsymbol{x}$, and so we expand the integral in (7.1) as a Taylor series:

$$\int \frac{Q'_{ij}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}\boldsymbol{x}' = |\boldsymbol{x}|^{-1}\int Q'_{ij}\,\mathrm{d}\boldsymbol{x}' + |\boldsymbol{x}|^{-3}\int (\boldsymbol{x}\cdot\boldsymbol{x}')\,Q'_{ij}\,\mathrm{d}\boldsymbol{x}'$$
$$-\tfrac{1}{2}|\boldsymbol{x}|^{-3}\int (\boldsymbol{x}')^2\,Q'_{ij}\,\mathrm{d}\boldsymbol{x}' + \tfrac{3}{2}|\boldsymbol{x}|^{-5}\int (\boldsymbol{x}\cdot\boldsymbol{x}')^2\,Q'_{ij}\,\mathrm{d}\boldsymbol{x}' + \dots .$$

Now the first term on the right is zero by virtue of continuity, while the second integral is zero for axisymmetric turbulence. This follows from the fact that, in axisymmetric turbulence, Q_{ij} may be written in the form

$$Q_{zz} = (A+2F)\,z^2 + C, \tag{7.2}$$
$$Q_{xx} = Ax^2 + B, \tag{7.3}$$
$$Q_{yy} = Ay^2 + B, \tag{7.4}$$
$$Q_{xz} = Q_{zx} = (A+F)\,xz, \tag{7.5}$$
$$Q_{yz} = Q_{zy} = (A+F)\,yz, \tag{7.6}$$
$$Q_{xy} = Q_{yx} = Axy, \tag{7.7}$$

where A, B, C and F are symmetric functions of $\boldsymbol{x}$. Combining the second and third integrals gives

$$\int \frac{Q'_{ij}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}\boldsymbol{x}' \approx \tfrac{1}{2}|\boldsymbol{x}|^{-5}\int [3(\boldsymbol{x}\cdot\boldsymbol{x}')^2 - \boldsymbol{x}^2(\boldsymbol{x}')^2]\,Q'_{ij}\,\mathrm{d}\boldsymbol{x}'.$$

For large $\boldsymbol{x}$, equation (7.1) now simplifies to

$$\frac{\partial Q_{ij}}{\partial t} = T_{ij}(\boldsymbol{u}, p) + \frac{1}{4\pi\tau}\frac{\partial^2}{\partial z^2}\left[|\boldsymbol{x}|^{-5}\int [3(\boldsymbol{x}\cdot\boldsymbol{x}')^2 - \boldsymbol{x}^2(\boldsymbol{x}')^2]\,Q'_{ij}\,\mathrm{d}\boldsymbol{x}'\right] + \text{HOT}. \tag{7.8}$$

Evidently, the contribution to $\partial Q_{ij}/\partial t$ from the Lorentz force is of order r^{-5} (at least at $t = 0$). This is the same as the contribution to $\partial Q_{ij}/\partial t$ from the pressure forces in conventional, homogeneous turbulence. It follows that Q_{ij} decays as r^{-5} immediately after $t = 0$, and it seems reasonable to suppose that Q_{ij} is of order r^{-5} for all $t > 0$. (Strictly we should evaluate all the time derivations at $t = 0$ if we wish to make this inference.)

We now turn our attention to the integrals I_1, I_2 and I_3. In particular, we wish to establish whether these converge. Let $Q_\parallel = Q_{zz}$ and $Q_\perp = Q_{xx} + Q_{yy}$. Then it is tedious, but not difficult to show that

$$\left.\begin{aligned}\frac{\partial Q_\parallel}{\partial t} &= T_\parallel(\boldsymbol{u}, p) - \frac{1}{8\pi\tau}\frac{\partial^4}{\partial z^4}\left[\frac{1}{|\boldsymbol{x}|}\right]\int (x'^2 + y'^2 - 2z'^2)\, Q'_\parallel \,\mathrm{d}\boldsymbol{x}' + \mathrm{HOT},\\ \frac{\partial Q_\perp}{\partial t} &= T_\perp(\boldsymbol{u}, p) - \frac{1}{8\pi\tau}\frac{\partial^4}{\partial z^4}\left[\frac{1}{|\boldsymbol{x}|}\right]\int (x'^2 + y'^2 - 2z'^2)\, Q'_\perp \,\mathrm{d}\boldsymbol{x}' + \mathrm{HOT}.\end{aligned}\right\} \tag{7.9}$$

To determine whether, immediately after $t = 0$, I_1, I_2 and I_3 converge it is necessary only to evaluate the leading terms associated with the Lorentz force. (We know that the pressure terms lead to convergent integrals, as in conventional turbulence, while the higher-order terms in the Lorentz force are of order r^{-6} and so have convergent integrals.) To show that I_1 and I_3 converge it is necessary for

$$\int_{|\boldsymbol{x}| > R_0} (x^2 + y^2)\frac{\partial^4}{\partial z^4}\left[\frac{1}{|\boldsymbol{x}|}\right]\mathrm{d}V$$

to be finite. To show that I_2 converges, on the other hand, we require

$$\int_{|\boldsymbol{x}| > R_0} z^2\frac{\partial^4}{\partial z^4}\left[\frac{1}{|\boldsymbol{x}|}\right]\mathrm{d}V$$

to be finite. In fact, it is readily confirmed that both integrals are zero. This is most easily established by writing the integrands as divergences and then integrating over a region $R_0 < |\boldsymbol{x}| < R_1$. Each volume integral then becomes the difference between two surface integrals which exactly cancel. (We omit the details as the algebra is routine.) It appears, then, that not only are I_1, I_2 and I_3 convergent, but that the contribution to these integrals from the long-range velocity correlations decays as r^{-1}. In short, the situation is precisely analogous to the effect of pressure on the integral moments I_{ijmn} in conventional, homogeneous turbulence.

Now we know from experiments on the decay of isotropic turbulence that the long-range velocity correlations are weak and do not greatly influence the decay. In our case we have additional long-range velocity correlations arising from the magnetic field. However, these appear to be no more potent than those arising from pressure forces in conventional homogeneous turbulence. Therefore, we shall tentatively ignore the influence of the long-range correlations on the decay of axisymmetric MHD turbulence. We cannot justify this, and indeed we know that it is strictly not valid even in conventional turbulence, as indicated by the slow evolution of Loitsyansky's integral. However, this does seem a reasonable starting point in what is otherwise an extremely complex problem.

8. The decay of MHD turbulence

We now explore the consequences of assuming that the long-range velocity correlations are weak. In particular, we use (6.15)–(6.17) to estimate the rate of growth of anisotropy and the decay of energy. First, however, we give a physical interpretation of these expressions. It is convenient to rewrite (6.15)–(6.17) in cylindrical coordinates using $\boldsymbol{x}$ as the position vector:

$$I_1(t) = -\int r^2 Q_\perp \,\mathrm{d}\boldsymbol{x} \approx I_1(0), \tag{8.1}$$

$$I_2(t) = -\int z^2 Q_\perp \,\mathrm{d}\boldsymbol{x} \approx I_2(0)\exp[-t/2\tau], \tag{8.2}$$

$$I_3(t) = -\int r^2 Q_\parallel \,\mathrm{d}\boldsymbol{x} \approx I_3(0)\exp[-t/2\tau]. \tag{8.3}$$

When N is small the turbulence decays much faster than the characteristic time τ, so that all three integrals are approximately constant during the decay. Let us turn, therefore, to the more interesting case of large N. Here the eddies elongate along the B-lines and, as the parallel integral scale grows, $Q_\perp$ and $Q_\parallel$ become progressively less dependent on z. It is clear then that the decay of I_2 is simply a manifestation of the growth of two-dimensional turbulence. This follows from the fact that, when $Q_\perp$ is independent of z,

$$I_2 \propto \int Q_\perp \,\mathrm{d}x\,\mathrm{d}y = 0.$$

(This integral is zero because of continuity.) Thus I_2 vanishes as the turbulence becomes two-dimensional.

The decline of I_3 is rather more difficult to interpret. At first glance it appears to suggest that u_z^2 falls off exponentially with time. However, this contradicts Moffatt's (1967) observation that u_z^2 actually increases. The results of §4 on the decay of transverse vortices offer an alternative explanation. It was shown there that such vortices develop a sheet-like structure consisting of thin interwoven layers of oppositely signed vorticity. This structure allows the angular momentum to decline exponentially on a time scale of τ. The 'sheets' are parallel to B and the direction of the dominant velocity, u_z oscillates back and forth across the sheet. Such a structure could allow the integral of $Q_\parallel$ to fall without u_z^2 becoming small, and this seems the most likely explanation for the decay of I_3.

Let us now introduce an energy equation for the turbulence. If E and D are the kinetic energy and Joule dissipation per unit mass, and ϵ is the viscous dissipation rate then

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\epsilon - D. \tag{8.4}$$

Equations (8.1) and (8.4) give us the estimates

$$E l_\perp^4 \, l_\parallel = \text{const}, \tag{8.5}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim -\frac{E^{3/2}}{l_\perp} - \left(\frac{l_\perp}{l_\parallel}\right)^2 \frac{E}{\tau}. \tag{8.6}$$

Here we have used Ohm's law to estimate D, and the idea of the energy cascade to estimate ϵ. Note that expression (8.5) plays a role in MHD turbulence which is quite analogous to that played by Loitsyansky's integral in isotropic turbulence. Equations (8.5) and (8.6) yield some familiar results in the limit of small and large N. When N is zero the two length scales $l_\parallel$ and $l_\perp$ are equal and we recover Kolmogorov's law:

$$E \sim E_0(1 + \alpha t/\tau_{t0})^{-10/7}. \tag{8.7}$$

Here α is a constant of order unity and τ_{t0} is the initial turnover time, $l_0/E_0^{1/2}$. When N is large, on the other hand, the Joule dissipation dominates and $l_\perp$ stays constant on a time scale of τ. Equations (8.5) and (8.6) become

$$El_\parallel = \text{const}, \tag{8.8}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim -\left(\frac{l_\perp}{l_\parallel}\right)^2 \frac{E}{\tau}, \tag{8.9}$$

which give

$$E \sim E_0(1+\beta t/\tau)^{-1/2}, \tag{8.10}$$

$$l_\parallel \sim l_0(1+\beta t/\tau)^{1/2}. \tag{8.11}$$

Here β is another constant of order one. Once again, these results are familiar. The first is Moffatt's (1967) prediction for the decay of energy at large N, while the second is Sommeria & Moreau's (1982) result for the rate of elongation of eddies. Interestingly, the isolated vortices analysed in §§4 and 5 also obey these laws. This is, perhaps, not surprising as the equations of motion for $N \gg 1$ are linear, and we could consider the evolution of the flow as comprising the superposition of many such isolated eddies each being acted upon by Lorentz forces. It should be noted, however, that (8.8)–(8.11) are valid only for a finite time. As noted in §6, the growth of $l_\parallel$ is such that the rate of fall of the Lorentz force is greater than the rate of fall of inertial forces, so that eventually, when $t \sim N_0^{4/3}\tau$, the removal of energy via the energy cascade becomes important.

It is reassuring that our new equation (8.5), which is the counterpart of Loitsyansky's integral for MHD turbulence, leads to results which are consistent with existing theories at low and high N. Equations (8.5) and (8.6) also provide an estimate of the effect of a small but finite Joule dissipation when N is small. Conversely, when N is large, but not so large as to prevent the breakup of vortices over one turnover time, they provide a first-order correction to (8.10). Let us introduce two scaled versions of t. At small N we use $t^* = t/\tau_{t0}$, and at large N we use $\hat{t} = t/\tau$, where $N = \tau_{t0}/\tau$ at $t = 0$. The first-order correction to (8.7) is readily shown to be of the form

$$E \sim E_0(1+\alpha t^*)^{-10/7}(1+\gamma_1 f(t^*)\,\hat{t})^{-1/2} + O(N^2),$$

while the first-order correction to (8.10) is

$$E \sim E_0(1+\beta\hat{t})^{-1/2}(1+\gamma_2 g(\hat{t})\,t^*)^{-10/7} + O(N^{-2}).$$

Here f and g are simple algebraic functions which satisfy $f(0) = g(0) = 1$ and take values between 0 and 1. These expressions may of course be matched for intermediate values of N by choosing $\gamma_1 = \beta$ and $\gamma_2 = \alpha$, although there is no physical justification for such a matching.

For the general case of arbitrary N equations (8.5) and (8.6) are not sufficient in themselves to predict the decay of energy. There are three unknowns $l_\perp$, $l_\parallel$ and E, so that a third equation, perhaps linking the length scales, is required. Nevertheless, the introduction of (8.5), as the counterpart of Loitsyansky's integral, seems a useful development.

9. Decay of MHD turbulence with $E \sim k^2$ spectra

So far we have assumed that the energy spectrum at low wavenumber is of the form $E \sim k^4$ However, Saffman (1967) has shown that, provided the net linear momentum of the fluid is non-zero, it is possible to have $E \sim k^2$ spectra. In such a case Loitsyansky's integral diverges as the far-field correlation Q_{ij} decays as r^{-3}. Integrals I_1, I_2 and I_3 likewise diverge. In this final section we indicate how our arguments might be extended to accommodate such a case. The discussion is brief. We start with an expression for the square of the global angular momentum in an open sphere of large radius:

$$\boldsymbol{H}^2 = \iint (\boldsymbol{x}\times\boldsymbol{u})\cdot(\boldsymbol{x}'\times\boldsymbol{u}')\,\mathrm{d}V'\,\mathrm{d}V.$$

This may be rewritten in the form

$$\boldsymbol{H}^2 = -\iint [(\boldsymbol{x}'\cdot\boldsymbol{u})(\boldsymbol{x}\cdot\boldsymbol{u}')-(\boldsymbol{x}\cdot\boldsymbol{x}')(\boldsymbol{u}\cdot\boldsymbol{u}')]\,\mathrm{d}V'\,\mathrm{d}V.$$

Next we introduce $\boldsymbol{r}=\boldsymbol{x}'-\boldsymbol{x}$, eliminate $\boldsymbol{x}'$, and ensemble average the resulting equation:

$$\boldsymbol{H}^2 = -\int x_i x_j \int \overline{u_i u_j'}\,\mathrm{d}V'\,\mathrm{d}V - \int x_i \int r_j \overline{u_i' u_j}\,\mathrm{d}V'\,\mathrm{d}V + \int x^2 \int \overline{u_i u_i'}\,\mathrm{d}V'\,\mathrm{d}V + \int x_i \int r_i \overline{u_j u_j'}\,\mathrm{d}V'\,\mathrm{d}V.$$

Following the argument given in §6, we now tentatively assume that the far-field velocity correlations are sufficiently weak for us to ignore those contributions to the inner integrals which lie outside the sphere. In that case

$$\boldsymbol{H}^2 = -\int x_i x_j\,\mathrm{d}V \int Q_{ij}\,dr - \int x_i\,\mathrm{d}V \int r_j Q_{ij}\,\mathrm{d}\boldsymbol{r} + \int \boldsymbol{x}^2\,\mathrm{d}V \int Q_{ii}\,\mathrm{d}\boldsymbol{r} + \int x_i\,\mathrm{d}V \int r_i Q_{jj}\,\mathrm{d}\boldsymbol{r} + O(u^2 l^4 R^4).$$

By symmetry, all terms on the right vanish except for the third, and so we obtain a simple relationship between the square of the angular momentum and the square of the net linear momentum:

$$\boldsymbol{H}^2 = \frac{2}{3}\int \boldsymbol{x}^2\,\mathrm{d}V \int Q_{ii}\,\mathrm{d}\boldsymbol{r}. \tag{9.1}$$

In a similar manner we may show that

$$\boldsymbol{H}_{\parallel}^2 = \int \boldsymbol{x}^2\,\mathrm{d}V\left[\frac{1}{3}\int Q_{\perp}\,\mathrm{d}\boldsymbol{r}\right], \tag{9.2}$$

$$\boldsymbol{H}_{\perp}^2 = \int \boldsymbol{x}^2\,\mathrm{d}V\left[\frac{2}{3}\int Q_{\parallel}\,\mathrm{d}\boldsymbol{r} + \frac{1}{3}\int Q_{\perp}\,\mathrm{d}\boldsymbol{r}\right]. \tag{9.3}$$

The permanence of $\boldsymbol{H}_{\parallel}$ and the decay of $\boldsymbol{H}_{\perp}$ now give relationships for the integrals of $Q_{\perp}$ and $Q_{\parallel}$. These are analogous to I_1, I_2 and I_3. (When the net linear momentum is zero, then (9.1) gives $\boldsymbol{H}^2$ as being of order $u^2 l^4 R^4$ or less, which is the source of the integrals considered in the preceeding sections.) Expression (9.2) in conjunction with conservation of $\boldsymbol{H}_{\parallel}$ suggests that

$$\boldsymbol{u}_{\perp}^2 l_{\perp}^2 l_{\parallel} = \text{const}, \tag{9.4}$$

during the decay of turbulence. This might be compared with (8.5). If (9.4) is coupled with energy equation (8.6), then we obtain $E \sim t^{-1.2}$ when N is small (Saffman's law) and $E \sim t^{-0.5}$ when N is large (which is the same as for $E \sim k^4$ turbulence).

10. Discussion

We have identified an integral, I_1, which is to MHD turbulence what Loitsyansky's integral is to isotropic turbulence. Just as Loitsyansky's integral varies with time, so we should not expect I_1 to be an invariant. However, if the long-range velocity correlations are weak, as in conventional turbulence, then I_1 should evolve only slowly. It then follows that $El_\perp^4 l_\parallel$ will be approximately constant as the turbulence decays. This assumption, coupled to an energy equation for the turbulence, leads to results which are consistent with existing theories, both at low N and high N.

Our picture of MHD turbulence at large values of N is essentially that described in §§4 and 5. Vortices parallel to $\boldsymbol{B}$ elongate along the field lines forming long columnar structures. These eddies grow as $l_\parallel \sim t^{1/2}$, while their energy decays as $E \sim t^{-1/2}$. Vortices normal to $\boldsymbol{B}$ distort into sheet-like structures, thus 'channelling' kinetic energy into the $u_\parallel^2$ component. At low N, on the other hand, we envisage a flow in which the dynamics are essentially independent of the Lorentz force, but in which Joule dissipation augments the destruction of the energy of the large scales.

Frequent comparisons are made between MHD turbulence and the pseudo-two-dimensional turbulence found in stratified or rotating media. Indeed, some authors point to MHD turbulence as a simple 'prototype' model of, for example, atmospheric turbulence. Our view is different. When the magnetic Reynolds number is small, there are quite fundamental distinctions between the Lorentz force on the one hand and Coriolis and buoyancy forces on the other. For example, stratified and rotating fluids propagate disturbances via internal waves, whereas disturbances in MHD turbulence propagate by pseudo-diffusion. Also, the Lorentz forces destroy those components of angular momentum which are normal to $\boldsymbol{B}$, while the Coriolis and buoyancy forces do not. Finally, the Lorentz force is dissipative, whereas the Coriolis and buoyancy forces do not alter the kinetic energy of the fluid. In short, the manner in which pseudo-two-dimensional turbulence is established is quite different in the two cases. In a rotating fluid, for example, the two-dimensional flow arises because disturbances propagate parallel to the rotation axis as internal waves, eventually forming Taylor columns. This is an inviscid process. In MHD turbulence, however, it is the need to conserve $\boldsymbol{H}_\parallel$ in the face of continual Joule dissipation that leads to an elongation of the eddies. Moreover, this distortion of the eddies is a diffusion process, with $l_\parallel$ growing like $t^{1/2}$, rather than wave-like propagation where $l_\parallel$ grows as t.

The author is indebted to two referees for their valuable comments. In particular, one referee has noted that one of the central results of the paper (equation (6.15)) may be obtained by alternative reasoning, i.e. by consideration of the shape of the spectrum at small wavenumber. This viewpoint is more in line with the conventional interpretation of the 'permanence of the big eddies' as being a consequence of the invariant shape of the spectrum at small k.

REFERENCES

Alemany, A., Moreau, R., Sulem, P. L. & Frisch, U. 1979 Influence of external magnetic field on homogeneous MHD turbulence. *J. Méc.* **18**, 280–313.

Batchelor, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

Batchelor, G. K. & Proudman, I. 1956 The large-scale structure of homogeneous turbulence. *Phil. Trans. R. Soc. Lond.* A **248**, 369–405.

Davidson, P. A. 1993 Similarities in the structure of swirling and buoyancy-driven flows. *J. Fluid Mech.* **252**, 357–382.

Davidson, P. A. 1995 Magnetic damping of jets and vortices. *J. Fluid Mech.* **299**, 153–186.

Hinze, J. O. 1975 *Turbulence*, 2nd edn. McGraw-Hill.

Japan Society of Mechanical Engineers 1988 *Visualised Flow*. Pergamon Press.

Jackson, J. D. 1962 *Classical Electrodynamics*. John Wiley & Sons.

Landau, L. D. &. Lifshitz, E. M. 1959 *Fluid Mechanics*, 1st edn. Pergamon Press.

Lesieur, M. 1987 *Turbulence in Fluids*. Kluwer.

Lumley, J. L. 1966 Invariants in turbulent flow. *Phys. Fluids* **9**, 2111–2113.

Moffatt, H. K. 1967 On the suppression of turbulence by a uniform magnetic field. *J. Fluid Mech.* **28**, 571–592.

Monin, A. S. & Yaglom, A. M. 1975 *Mechanics of Turbulence*, vol. 2. MIT Press.

Moreau, R. 1990 *Magnetohydrodynamics*. Kluwer.

Saffman, P. G. 1967 The large-scale structure of homogeneous turbulence. *J. Fluid Mech.* **27**, 581–593.

Shercliff, J. A. 1965 *A Textbook of Magnetohydrodynamics*. Pergamon Press.

Sommeria, J. & Moreau, R. 1982 Why, how, and when, MHD turbulence becomes two-dimensional. *J. Fluid Mech.* **118**, 507–518.